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OPTIMAL CONSUMPTION AND INVESTMENT PROBLEM WITH REGIME-SWITCHING AND CARA UTILITY

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Abstract. We use the dynamic programming method to investigate the optimal consumption and investment problem with regime-switching. We derive the optimal solutions in closed-form with constant absolute risk aversion (CARA) utility.

1. Introduction

After the pioneer work of Merton [3, 4], portfolio optimization problem has been one of the most important and active area in mathematical finance. Recently, portfolio selection problems combined with regime-switching are widely considered. ([1, 6, 5])

In this paper we investigate the optimal consumption and investment problem with regime-switching under the dynamic programming framework of Karatzas *et al.* [2]. We use the constant absolute risk aversion (CARA) utility function to obtain the optimal solutions in closed-form.

2. The financial market

Let us define a standard Brownian motion B_t and a continuous-time two-state Markov chain ϵ_t on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that B_t and ϵ_t are independent and the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is generated by both the Brownian motion B_t and the Markov chain ϵ_t.

We assume that two assets are traded in the financial market: One is a riskless asset with constant interest rate r and the other is a risky asset. We assume that there are two regimes 1, 2 in the market and regime i switches into regime j at the first jump time of an independent

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Poisson process with intensity λ_i, for $i, j \in \{1, 2\}$. Here two states of regimes 1, 2 are economically considered as bull market and bear market. So, in regime $i \in \{1, 2\}$, the risky asset price process follows $dS_t/S_t = \mu_i dt + \sigma_i dB_t$. The market price of risk is defined by $\theta_i \triangleq (\mu_i - r)/\sigma_i,\ i = 1,\ 2$. Let π_t be the $\mathcal{F}_t$-progressively measurable portfolio process, the dollar-amount of the agent's wealth invested in the risky asset at time t and c_t be the nonnegative $\mathcal{F}_t$-progressively measurable consumption process at time t. We assume that they hold the following technical conditions:

$$\int_0^t c_s ds < \infty \quad \text{and} \quad \int_0^t \pi_s^2 ds < \infty, \text{ for all } t \geq 0, \text{ almost surely (a.s.)}.$$

The agent's wealth process X_t at time t follows

$$dX_t = [rX_t + \pi_t(\mu_i - r) - c_t]\, dt + \sigma_i \pi_t dB_t, \quad X_0 = x > 0,\ i = 1,\ 2.$$

3. The optimization problem

The agent's optimization problem with CARA utility $u(c) := -e^{-\gamma c}/\gamma$, is to maximize her expected utility

$$V_i(x) = \sup_{(c,\pi)\in\mathcal{A}(x)} \mathbb{E}\left[-\int_0^{\tau_i} \frac{e^{-\rho t - \gamma c_t}}{\gamma} dt + e^{-\rho\tau_i} V_j(X_{\tau_i})\right], \tag{3.1}$$

where τ_i is the first jump time since the beginning of regime i, $\rho > 0$ is a subjective discount rate, $\gamma > 0$ is absolute risk aversion and $\mathcal{A}(x)$ is an admissible set of pairs (c, π) at x, where $i, j \in \{1, 2\}$ and $i \neq j$.

Assumption 3.1.

$$\rho - r + \frac{\theta_i^2}{2} > 0,\ i \in \{1, 2\}.$$

Next theorem gives our main results.

Theorem 3.2. *The value function is given by*

$$V_i(x) = -\frac{M_i}{\gamma r} e^{-\gamma r x}, \tag{3.2}$$

where M_i and M_j are the solutions of the system of algebraic equations

$$\left(\rho + \lambda_i - r + \frac{\theta_i^2}{2}\right) M_i + rM_i \log M_i - \lambda_i M_j = 0, \tag{3.3}$$

where $i, j \in \{1, 2\}$ and $i \neq j$, and the optimal policies (c_i^*, π_i^*) are given by

$$c_i^* = rx - \frac{1}{\gamma}\log M_i \quad \text{and} \quad \pi_i^* = \frac{\theta_i}{\gamma r \sigma_i},\ i = 1,\ 2. \tag{3.4}$$

Proof. From the optimization problem (3.1), we have the following coupled Bellman equations

$$\begin{aligned}\max_{(c_i,\pi_i)} \Bigg[&\{rx + \pi_i(\mu_i - r) - c_i\} V_i'(x) + \frac{1}{2}\sigma_i^2 \pi_i^2 V_i''(x) \\ &-(\rho + \lambda_i)V_i(x) + \lambda_i V_j(x) - \frac{e^{-\gamma c_i}}{\gamma} \Bigg] = 0,\end{aligned} \tag{3.5}$$

where $i, j \in \{1, 2\}$ and $i \neq j$. The first-order conditions (FOCs) imply

$$c_i^* = -\frac{1}{\gamma}\log(V_i'(x)) \quad \text{and} \quad \pi_i^* = -\frac{\theta_i V_i'(x)}{\sigma_i V_i''(x)},\ i = 1,\ 2. \tag{3.6}$$

We assume that the optimal consumption $c_i^* = C_i(x),\ i = 1,\ 2$, is a function of wealth. And let $X_i(\cdot),\ i = 1,\ 2$, be the inverse function of $C_i(\cdot),\ i = 1,\ 2$. That is, $X_i(\cdot) = C_i^{-1}(\cdot),\ i = 1,\ 2$. Then the FOCs (3.6) imply

$$V_i'(x) = e^{-\gamma C_i(x)} \quad \text{and} \quad V_i''(x) = -\frac{\gamma}{X_i'(c_i)} e^{-\gamma C_i(x)},\ i = 1,\ 2. \tag{3.7}$$

Plugging the FOCs (3.6) with (3.7) into the Bellman equation (3.5), then we obtain

$$\begin{aligned} &rX_i(c_i)e^{-\gamma c_i} + \frac{\theta_i^2}{2\gamma} X_i'(c_i) e^{-\gamma c_i} - c_i e^{-\gamma c_i} \\ &- (\rho + \lambda_i)V_i(X_i(c_i)) + \lambda_i V_j(X_i(c_i)) - \frac{e^{-\gamma c_i}}{\gamma} = 0,\end{aligned} \tag{3.8}$$

where $i, j \in \{1, 2\}$ and $i \neq j$. Taking derivative of the equation (3.8) with respect to c_i, then we obtain

$$\begin{aligned} rX_i'(c_i)e^{-\gamma c_i} - \gamma r X_i(c_i) e^{-\gamma c_i} + \frac{\theta_i^2}{2\gamma} X_i''(c_i) e^{-\gamma c_i} - \frac{\theta_i^2}{2} X_i'(c_i) e^{-\gamma c_i} \\ + \gamma c_i e^{-\gamma c_i} - (\rho + \lambda_i) X_i'(c_i) e^{-\gamma c_i} + \lambda_i X_i'(c_i) e^{-\gamma c_j} = 0,\end{aligned}$$

where $i, j \in \{1, 2\}$ and $i \neq j$, since $x = X_1(c_1) = X_2(c_2)$ implies $V_j'(X_i(c_i)) = V_j'(x) = e^{-\gamma c_j}$, where $i, j \in \{1, 2\}$ and $i \neq j$. Thus we obtain the coupled second order differential equations with respect to c_1 and c_2

$$\begin{aligned}(3.9)\qquad &\frac{\theta_i^2}{2\gamma}X_i''(c_i) - \left(\rho + \lambda_i - r + \frac{\theta_i^2}{2}\right)X_i'(c_i)\\ &- \gamma r X_i(c_i) + \gamma c_i + \lambda_i e^{-\gamma(c_j - c_i)}X_i'(c_i) = 0,\end{aligned}$$

where $i, j \in \{1, 2\}$ and $i \neq j$. If we conjecture the solution $X_i(c_i)$ of the form

$$(3.10)\qquad X_i(c_i) = \frac{c_i}{r} + \frac{1}{\gamma r}\log M_i \quad \text{and} \quad c_i = rx - \frac{1}{\gamma}\log M_i,\ i = 1,\ 2,$$

for some constant $M_i > 0$, then $X_i'(c_i) = 1/r$ and $X_i''(c_i) = 0,\ i = 1,\ 2.$ The equation (3.10) implies

$$c_j - c_i = \frac{1}{\gamma}\log\frac{M_i}{M_j},$$

where $i, j \in \{1, 2\}$ and $i \neq j$. So the equation (3.9) can be reduced into the system of algebraic equations (3.3).

Now we want to show that there exists a unique pair solution (M_1, M_2) to (3.3). Without loss of generality, we may assume that $\theta_i < \theta_j$. Let

$$M_j = f(M_i) := \frac{1}{\lambda_i}\left(\rho + \lambda_i - r + \frac{\theta_i^2}{2}\right)M_i + \frac{r}{\lambda_i}M_i\log M_i > 0,$$

for $M_i > 0$, and let

$$f_1(M_i) := \frac{1}{\lambda_i}\left(\rho + \lambda_i - r + \frac{\theta_i^2}{2}\right) + \frac{r}{\lambda_i}\log M_i > 0,$$

then $f_1'(M_i) = r/(\lambda_i M_i) > 0$. Now we consider the constants $\bar{x}$ and $\underline{x}$ with $\bar{x} > \underline{x}$ as follows:

$$\bar{x} := e^{-\frac{1}{r}\left(\rho - r + \frac{\theta_i^2}{2}\right)} < 1 \quad \text{and} \quad \underline{x} := e^{-\frac{1}{r}\left(\rho + \lambda_i - r + \frac{\theta_i^2}{2}\right)} < 1,$$

where the inequalities are obtained from Assumption 3.1. Then $f_1(\bar{x}) = 1,\ f_1(\underline{x}) = 0$. Since $f_1(M_i) > 0$, $M_i > \underline{x}$.

Now let

$$g(M_i) := \left(\rho + \lambda_j - r + \frac{\theta_j^2}{2}\right)M_i f_1(M_i) + rM_i f_1(M_i)\log M_i f_1(M_i) - \lambda_j M_i,$$

and let

$$g_1(M_i) := \left(\rho + \lambda_j - r + \frac{\theta_j^2}{2}\right)f_1(M_i) + rf_1(M_i)\log M_i f_1(M_i) - \lambda_j.$$

Note that $g_1(\bar{x}) = (\theta_j^2 - \theta_i^2)/2 > 0$. Since, by l'Hospital's rule,

$$\lim_{M_i \to \underline{x}+} \frac{\log M_i f_1(M_i)}{1/f_1(M_i)} = \lim_{M_i \to \underline{x}+} \frac{f_1(M_i)(f_1(M_i) + M_i f_1'(M_i))}{-M_i f_1'(M_i)} = 0,$$

$\lim_{M_i \to \underline{x}+} g_1(M_i) = -\lambda_j < 0$. Thus, by intermediate value theorem, there exists $\bar{M} > 0$ such that $g_1(\bar{M}) = 0$ and $\underline{x} < \bar{M} < \bar{x}$. Taking derivative of $g_1(M_i)$, then

$$\begin{aligned} & g_1'(M_i) \\ & = \left(\rho + \lambda_j + \frac{\theta_j^2}{2}\right) f_1'(M_i) + r f_1'(M_i) \log M_i f_1(M_i) + r \frac{f_1(M_i)}{M_i} \\ & = \frac{r}{\lambda_i M_i} h(M_i), \end{aligned} \tag{3.11}$$

where

$$h(M_i) := \left(2\rho - r + \lambda_i + \lambda_j + \frac{\theta_i^2 + \theta_j^2}{2}\right) + r \log M_i^2 f_1(M_i).$$

Taking derivative of $h(M_i)$, then

$$h'(M_i) = r\left(\frac{2}{M_i} + \frac{f_1'(M_i)}{f_1(M_i)}\right) > 0.$$

Thus $h(M_i)$ is increasing, $\lim_{M_i \to \underline{x}+} h(M_i) = -\infty$ and $h(\bar{x}) = r + \lambda_i + \lambda_j + (\theta_j^2 - \theta_i^2)/2 > 0$. Again, by intermediate value theorem, there exists a unique $x^* > 0$ such that $h(x^*) = 0$ and $\underline{x} < x^* < \bar{x}$. That is, $h(M_1) < 0$ for $(\underline{x}, x^*)$ and $h(M_1) > 0$ for (x^*, ∞). This implies $g_1'(M_1) < 0$ for $(\underline{x}, x^*)$ and $g_1'(M_1) > 0$ for (x^*, ∞). Thus $g_1(M_1)$ is decreasing and negative for $(\underline{x}, x^*)$ and $g_1(M_1)$ is increasing for (x^*, ∞). Therefore $\bar{M}$ with $x^* < \bar{M} < \bar{x}$ is the unique solution to $g_1(M_i) = 0$, and consequently we obtain the unique pair solution (M_1, M_2) to (3.3).

Plugging c_i in (3.10) into (3.7), then we derive

$$V_i'(x) = M_i e^{-\gamma r x} \quad \text{and} \quad V_i''(x) = -\gamma r M_i e^{-\gamma r x}. \tag{3.12}$$

Also plugging (3.12) into the FOCs (3.6), then we obtain the optimal policies (c_i^*, π_i^*) in (3.4). Therefore, from the Bellman equations (3.5), we obtain the value function $V_i(x)$ in (3.2). □

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